

Real-Time Rendering

Fourth Edition

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Appendix A

Some Linear Algebra

BOOK I. DEFINITIONS.

A point is that which has no part.

A line is a breadthless length.

The extremities of a line are points.

A straight line is a line which lies evenly with the points on itself.

—The first four definitions from *Elements* by Euclid [4]

This appendix deals with the fundamental concepts of linear algebra that are of greatest use for computer graphics. Our presentation will not be as mathematically abstract and general as these kinds of descriptions often are, but will rather concentrate on what is most relevant. For the inexperienced reader, it can be seen as a short introduction to the topic, and may otherwise serve as a review.

We will start with an introduction to the Euclidean spaces. This may feel abstract at first, but in the section that follows, these concepts are connected to geometry, bases, and matrices. So, bite the bullet during the first section, and reap the rewards in the rest of the appendix and in many of the other chapters of this book.

A.1 Euclidean Space

The n -dimensional real Euclidean space is denoted \mathbb{R}^n . A vector \mathbf{v} in this space is an n -tuple, that is, an ordered list of real numbers:

$$\mathbf{v} \in \mathbb{R}^n \iff \mathbf{v} = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \text{ with } v_i \in \mathbb{R}, i = 0, \dots, n-1. \quad (\text{A.1})$$

Note that the subscripts start at 0 and end at $n-1$, a numbering system that follows the indexing of arrays in many programming languages, such as C and C++.

This makes it easier to convert from formula to code. Some computer graphics books and linear algebra books start at 1 and end at n .

The vector can also be presented as a row vector, but most computer graphics texts use column vectors, in what is called the column-major form. We call v_0, \dots, v_{n-1} the elements, the coefficients, or the components of the vector \mathbf{v} . All bold lowercase letters are vectors that belong to \mathbb{R}^n , and italicized lowercase letters are scalars that belong to \mathbb{R} . As an example, a two-dimensional vector is $\mathbf{v} = (v_0, v_1)^T \in \mathbb{R}^2$. For vectors in a Euclidean space there exist two operators, *addition* and *multiplication by a scalar*, which work as might be expected:

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{pmatrix} + \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} u_0 + v_0 \\ u_1 + v_1 \\ \vdots \\ u_{n-1} + v_{n-1} \end{pmatrix} \in \mathbb{R}^n \quad \text{(addition)} \quad (\text{A.2})$$

and

$$a\mathbf{u} = \begin{pmatrix} au_0 \\ au_1 \\ \vdots \\ au_{n-1} \end{pmatrix} \in \mathbb{R}^n \quad \text{(multiplication by a scalar)}. \quad (\text{A.3})$$

The “ $\in \mathbb{R}^n$ ” simply means that addition and multiplication by a scalar yields vectors of the same space. As can be seen, addition is done componentwise, and multiplication is done by multiplying all elements in the vector with the scalar a .

A series of rules hold for Euclidean space—in fact, these *are* the definition of Euclidean space. Addition of vectors in a Euclidean space also works as might be expected:

$$\begin{aligned} (i) \quad & (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \quad \text{(associativity)} \\ (ii) \quad & \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad \text{(commutativity)}. \end{aligned} \quad (\text{A.4})$$

There is a unique vector, called the zero vector, which is $\mathbf{0} = (0, 0, \dots, 0)$ with n zeros, such that

$$(iii) \quad \mathbf{0} + \mathbf{v} = \mathbf{v} \quad \text{(zero identity)}. \quad (\text{A.5})$$

There is also a unique vector $-\mathbf{v} = (-v_0, -v_1, \dots, -v_{n-1})$ such that

$$(iv) \quad \mathbf{v} + (-\mathbf{v}) = \mathbf{0} \quad \text{(additive inverse)}. \quad (\text{A.6})$$

Rules for multiplication by a scalar work as follows:

$$\begin{aligned} (i) \quad & (ab)\mathbf{u} = a(b\mathbf{u}) \quad \text{(associativity)} \\ (ii) \quad & (a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u} \quad \text{(distributivity)} \\ (iii) \quad & a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v} \quad \text{(distributivity)} \\ (iv) \quad & 1\mathbf{u} = \mathbf{u} \quad \text{(multiplicative one)}. \end{aligned} \quad (\text{A.7})$$

For a Euclidean space we may also compute the *dot product* of two vectors \mathbf{u} and \mathbf{v} . This value is also called *inner (dot) product* or *scalar product*. The dot product is denoted $\mathbf{u} \cdot \mathbf{v}$, and its definition is shown below:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \phi \quad (\text{dot product}), \quad (\text{A.8})$$

where $\mathbf{u} \cdot \mathbf{v} = 0$ if $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$. Here, ϕ (shown at the left in Figure A.3) is the smallest angle between \mathbf{u} and \mathbf{v} . Several conclusions can be drawn from the sign of the dot product, assuming that both vectors have non-zero length. First, $\mathbf{u} \cdot \mathbf{v} = 0 \Leftrightarrow \mathbf{u} \perp \mathbf{v}$, i.e., \mathbf{u} and \mathbf{v} are orthogonal (perpendicular) if their dot product is zero. Second, if $\mathbf{u} \cdot \mathbf{v} > 0$, then it can be seen that $0 \leq \phi < \frac{\pi}{2}$, and likewise if $\mathbf{u} \cdot \mathbf{v} < 0$ then $\frac{\pi}{2} < \phi \leq \pi$. For the dot product we have the rules:

$$\begin{aligned} (i) \quad & \mathbf{u} \cdot \mathbf{u} \geq 0, \text{ with } \mathbf{u} \cdot \mathbf{u} = 0 \text{ if and only if } \mathbf{u} = (0, 0, \dots, 0) = \mathbf{0} \\ (ii) \quad & (\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} \quad (\text{distributivity}) \\ (iii) \quad & (a\mathbf{u}) \cdot \mathbf{v} = a(\mathbf{u} \cdot \mathbf{v}) \quad (\text{associativity}) \\ (iv) \quad & \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \quad (\text{commutativity}) \\ (v) \quad & \mathbf{u} \cdot \mathbf{v} = 0 \iff \mathbf{u} \perp \mathbf{v}. \end{aligned} \quad (\text{A.9})$$

The last formula means that if the dot product is zero then the vectors are orthogonal (perpendicular).

A.2 Geometrical Interpretation

Here, we will interpret the vectors (from the previous section) geometrically. For this, we first need to introduce the concepts of *linear independence* and the *basis*. If the only scalars to satisfy Equation A.10 below are $v_0 = v_1 = \dots = v_{n-1} = 0$, then the vectors, $\mathbf{u}_0, \dots, \mathbf{u}_{n-1}$, are said to be linearly independent. Otherwise, the vectors are linearly dependent:

$$v_0 \mathbf{u}_0 + \dots + v_{n-1} \mathbf{u}_{n-1} = \mathbf{0} \quad (\text{A.10})$$

For example, the vectors $\mathbf{u}_0 = (4, 3)$ and $\mathbf{u}_1 = (8, 6)$ are *not* linearly independent, since $v_0 = 2$ and $v_1 = -1$ (among others) satisfy Equation A.10. Only vectors that are parallel are linearly dependent.

If a set of n vectors, $\mathbf{u}_0, \dots, \mathbf{u}_{n-1} \in \mathbb{R}^n$, is linearly independent and any vector $\mathbf{v} \in \mathbb{R}^n$ can be written as

$$\mathbf{v} = \sum_{i=0}^{n-1} v_i \mathbf{u}_i, \quad (\text{A.11})$$

then the vectors $\mathbf{u}_0, \dots, \mathbf{u}_{n-1}$ are said to span Euclidean space \mathbb{R}^n . If, in addition, v_0, \dots, v_{n-1} are uniquely determined by \mathbf{v} for all $\mathbf{v} \in \mathbb{R}^n$, then $\mathbf{u}_0, \dots, \mathbf{u}_{n-1}$ is called a basis of \mathbb{R}^n . What this means is that every vector can be described uniquely by n scalars (v_0, v_1, \dots, v_{n-1}) and the basis vectors $\mathbf{u}_0, \dots, \mathbf{u}_{n-1}$. The dimension of the space is n , if n is the largest number of linearly independent vectors in the space.

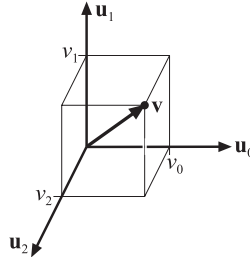


Figure A.1. A three-dimensional vector $\mathbf{v} = (v_0, v_1, v_2)$ expressed in the basis formed by $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2$ in \mathbb{R}^3 . Note that this is a right-handed system.

An example of a linearly independent basis is $\mathbf{u}_0 = (4, 3)$ and $\mathbf{u}_1 = (2, 6)$. This spans \mathbb{R}^2 , as any vector can be expressed as a unique combination of these two vectors. For example, $(-5, -6)$ is described by $v_0 = -1$ and $v_1 = -0.5$ and no other combinations of v_0 and v_1 .

To completely describe a vector, \mathbf{v} , one should use Equation A.11, that is, using both the vector components, v_i and the basis vectors, \mathbf{u}_i . That often becomes impractical, and therefore the basis vectors are often omitted in mathematical manipulation when the same basis is used for all vectors. If that is the case, then \mathbf{v} can be described as

$$\mathbf{v} = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}, \quad (\text{A.12})$$

which is exactly the same vector description as in Expression A.1, and so this is the one-to-one mapping of the vectors in Section A.1 onto geometrical vectors. An illustration of a three-dimensional vector is shown in Figure A.1. In this book, a vector \mathbf{v} can be thought of as a point in space or as a directed line segment (i.e., a direction vector). All rules from Section A.1 apply in the geometrical sense, too. For example, the addition and the scaling operators from Equation A.2 are visualized in Figure A.2. A basis can also have different “handedness.”

A three-dimensional, right-handed basis is one in which the x -axis is along the thumb, the y -axis is along index-finger, and the z -axis is along the middle finger. If this is done with the left hand, a left-handed basis is obtained. See page 11 for a more formal definition of “handedness.”

Now we will go back to the study of basis for a while, and introduce a special kind of basis that is said to be *orthonormal*. For such a basis, consisting of the basis vectors $\mathbf{u}_0, \dots, \mathbf{u}_{n-1}$, the following must hold:

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases} \quad (\text{A.13})$$

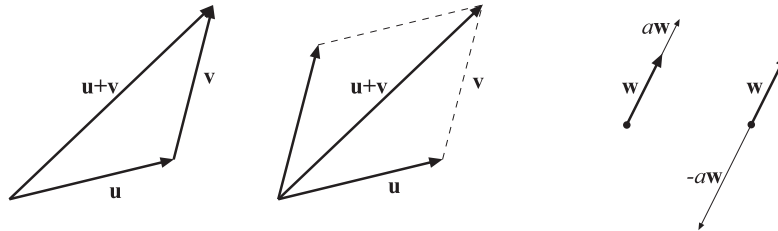


Figure A.2. Vector-vector addition is shown in the two figures on the left. They are called the head-to-tail axiom and the parallelogram rule. The two rightmost figures show scalar-vector multiplication for a positive and a negative scalar, a and $-a$, respectively.

This means that every basis vector must have a length of one, i.e., $\|\mathbf{u}_i\| = 1$, and that each pair of basis vectors must be orthogonal, i.e., the angle between them must be $\pi/2$ radians (90°). In this book, we mostly use two- and three-dimensional orthonormal bases. If the basis vectors are mutually perpendicular, but not of unit length, then the basis is called *orthogonal*. Orthonormal bases do not have to consist of simple vectors. For example, in precomputed radiance transfer techniques the bases often are either spherical harmonics or wavelets. In general, the vectors are exchanged for functions, and the dot product is augmented to work on these functions. Once that is done, the concept of orthonormality applies as above.

For an orthonormal basis, the dot product (Equation A.8) can be simplified to:

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=0}^{n-1} u_i v_i \quad (\text{dot product}). \quad (\text{A.14})$$

Let $\mathbf{p} = (p_0, \dots, p_{n-1})$, then for an orthonormal basis it can also be shown that $p_i = \mathbf{p} \cdot \mathbf{u}_i$. This means that if you have a vector \mathbf{p} and a basis (with the basis vectors $\mathbf{u}_0, \dots, \mathbf{u}_{n-1}$), then you can easily get the elements of that vector in that basis by taking the dot product between the vector and each of the basis vectors. The most common basis is called the standard basis, where the basis vectors are denoted \mathbf{e}_i . The i th basis vector has zeros everywhere except in position i , which holds a one. For three dimensions, this means $\mathbf{e}_0 = (1, 0, 0)$, $\mathbf{e}_1 = (0, 1, 0)$, and $\mathbf{e}_2 = (0, 0, 1)$. We also denote these vectors \mathbf{e}_x , \mathbf{e}_y , and \mathbf{e}_z , since they are what we normally call the x -, the y -, and the z -axes.

The norm of a vector, denoted $\|\mathbf{u}\|$, is a non-negative number that can be expressed using the dot product:

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{\left(\sum_{i=0}^{n-1} u_i^2\right)} \quad (\text{norm}). \quad (\text{A.15})$$

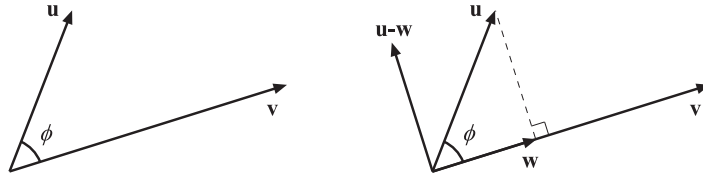


Figure A.3. The left figure shows the notation and geometric situation for the dot product. In the rightmost figure, orthographic projection is shown. The vector \mathbf{u} is orthogonally (perpendicularly) projected onto \mathbf{v} to yield \mathbf{w} .

However, this is only true for an orthonormal basis. For the norm, we have:

$$\begin{aligned}
 (i) \quad & \|\mathbf{u}\| = 0 \iff \mathbf{u} = (0, 0, \dots, 0) = \mathbf{0} \\
 (ii) \quad & \|a\mathbf{u}\| = |a| \|\mathbf{u}\| \\
 (iii) \quad & \|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\| \quad (\text{triangle inequality}) \\
 (iv) \quad & |\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\| \quad (\text{Cauchy-Schwartz inequality}).
 \end{aligned} \tag{A.16}$$

The norm of a vector (see Equation A.15) can be thought of as the length of the vector. For example, the length of a two-dimensional vector, \mathbf{u} , is $\|\mathbf{u}\| = \sqrt{u_0^2 + u_1^2}$, is computed using the Pythagorean theorem. To create a vector of unit length, i.e., of length one, the vector has to be normalized. This can be done by dividing by the length of the vector: $\mathbf{q} = \frac{1}{\|\mathbf{p}\|} \mathbf{p}$, where \mathbf{q} is the normalized vector, which also is called a unit vector.

A useful property of the dot product is that it can be used to project a vector orthogonally onto another vector. This orthogonal projection (vector), \mathbf{w} , of a vector \mathbf{u} onto a vector \mathbf{v} is depicted on the right in Figure A.3. For arbitrary vectors \mathbf{u} and \mathbf{v} , \mathbf{w} is determined by

$$\mathbf{w} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = t\mathbf{v}, \tag{A.17}$$

where t is a scalar. The reader is encouraged to verify that Expression A.17 is indeed correct, which is done by inspection and the use of Equation A.8. The projection also gives us an orthogonal decomposition of \mathbf{u} , which is divided into two parts, \mathbf{w} and $(\mathbf{u} - \mathbf{w})$. It can be shown that $\mathbf{w} \perp (\mathbf{u} - \mathbf{w})$, and of course $\mathbf{u} = \mathbf{w} + (\mathbf{u} - \mathbf{w})$ holds. An additional observation is that if \mathbf{v} is normalized, then the projection is $\mathbf{w} = (\mathbf{u} \cdot \mathbf{v})\mathbf{v}$. This means that $\|\mathbf{w}\| = |\mathbf{u} \cdot \mathbf{v}|$, i.e., the length of \mathbf{w} is the absolute value of the dot product between \mathbf{u} and \mathbf{v} .

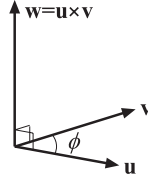


Figure A.4. The geometry involved in the cross product.

Cross Product

The cross product, also called the vector product, is another essential vector operation, second only to the dot product in its significance for computer graphics.

The cross product in \mathbb{R}^3 of two vectors, \mathbf{u} and \mathbf{v} , denoted by $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, is defined by a unique vector \mathbf{w} with the following properties:

- $\|\mathbf{w}\| = \|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \phi$, where ϕ is, again, the smallest angle between \mathbf{u} and \mathbf{v} . See Figure A.4.
- $\mathbf{w} \perp \mathbf{u}$ and $\mathbf{w} \perp \mathbf{v}$.
- \mathbf{u} , \mathbf{v} , \mathbf{w} form a right-handed system if the basis is right-handed and a left-handed system otherwise.

From this definition, it is deduced that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u} \parallel \mathbf{v}$ (i.e., \mathbf{u} and \mathbf{v} are parallel), since then $\sin \phi = 0$. The cross product also comes equipped with the following laws of calculation, where \mathbf{w} is an arbitrary vector:

$$\begin{aligned}
 (i) \quad & \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} && \text{(anti-commutativity)} \\
 (ii) \quad & (a\mathbf{u} + b\mathbf{v}) \times \mathbf{w} = a(\mathbf{u} \times \mathbf{w}) + b(\mathbf{v} \times \mathbf{w}) && \text{(linearity)} \\
 (iii) \quad & \left. \begin{aligned} & (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u} \\ & = (\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{w} \\ & = -(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v} = -(\mathbf{w} \times \mathbf{v}) \cdot \mathbf{u} \end{aligned} \right\} && \text{(scalar triple product)} \quad (\text{A.18}) \\
 (iv) \quad & \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} && \text{(vector triple product)}
 \end{aligned}$$

From these laws, it is obvious that the order of the operands is crucial in getting correct results from the calculations.

For three-dimensional vectors, \mathbf{u} and \mathbf{v} , in an orthonormal basis, the cross product is computed according to Equation A.19:

$$\mathbf{w} = \begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix} = \mathbf{u} \times \mathbf{v} = \begin{pmatrix} u_y v_z - u_z v_y \\ u_z v_x - u_x v_z \\ u_x v_y - u_y v_x \end{pmatrix}. \quad (\text{A.19})$$

A method called Sarrus's scheme, which is simple to remember, can be used to derive this formula:

$$\begin{array}{ccccccc}
 & + & + & + & & - & - & - \\
 & \searrow & \searrow & \searrow & & \swarrow & \swarrow & \swarrow \\
 & \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z & \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z & \\
 & u_x & u_y & u_z & u_x & u_y & u_z & \\
 & v_x & v_y & v_z & v_x & v_y & v_z &
 \end{array} \tag{A.20}$$

To use the scheme, follow the diagonal arrows, and for each arrow, generate a term by multiplying the elements along the direction of the arrow and giving the product the sign associated with that arrow. The result is shown below and it is exactly the same formula as in Equation A.19, as expected:

$$\mathbf{u} \times \mathbf{v} = +\mathbf{e}_x(u_y v_z) + \mathbf{e}_y(u_z v_x) + \mathbf{e}_z(u_x v_y) - \mathbf{e}_x(u_z v_y) - \mathbf{e}_y(u_x v_z) - \mathbf{e}_z(u_y v_x).$$

A.3 Matrices

This section presents the definitions concerning matrices and some common, useful operations on them. Even though this presentation is (mostly) for arbitrarily sized matrices, square matrices of the sizes 2×2 , 3×3 , and 4×4 will be used in the chapters of this book. Note that Chapter 4.5 deals with transforms represented by matrices.

A.3.1 Definitions and Operations

A matrix, \mathbf{M} , can be used as a tool for manipulating vectors and points. \mathbf{M} is described by $p \times q$ scalars (complex numbers are an alternative, but not relevant here), m_{ij} , $0 \leq i \leq p-1$, $0 \leq j \leq q-1$, ordered in a rectangular fashion (with p rows and q columns) as shown in Equation A.21:

$$\mathbf{M} = \begin{pmatrix} m_{00} & m_{01} & \cdots & m_{0,q-1} \\ m_{10} & m_{11} & \cdots & m_{1,q-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{p-1,0} & m_{p-1,1} & \cdots & m_{p-1,q-1} \end{pmatrix} = [m_{ij}]. \tag{A.21}$$

The notation $[m_{ij}]$ will be used in the equations below and is merely a shorter way of describing a matrix. There is a special matrix called the *unit matrix*, \mathbf{I} , which is square and contains ones in the diagonal and zeros elsewhere. This is also called the

identity matrix. Equation A.22 shows its general appearance. This is the matrix-form counterpart of the scalar number one:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}. \quad (\text{A.22})$$

Next, the most ordinary operations on matrices will be reviewed.

Matrix-Matrix Addition

Adding two matrices, say \mathbf{M} and \mathbf{N} , is possible only for equal-sized matrices and is defined as

$$\mathbf{M} + \mathbf{N} = [m_{ij}] + [n_{ij}] = [m_{ij} + n_{ij}], \quad (\text{A.23})$$

that is, componentwise addition, similar to vector-vector addition. The resulting matrix is of the same size as the operands. The following operations are valid for matrix-matrix addition: *i*) $(\mathbf{L} + \mathbf{M}) + \mathbf{N} = \mathbf{L} + (\mathbf{M} + \mathbf{N})$, *ii*) $\mathbf{M} + \mathbf{N} = \mathbf{N} + \mathbf{M}$, *iii*) $\mathbf{M} + \mathbf{0} = \mathbf{M}$, *iv*) $\mathbf{M} - \mathbf{M} = \mathbf{0}$, which are all easy to prove. Note that $\mathbf{0}$ is a matrix containing only zeros.

Scalar-Matrix Multiplication

A scalar a and a matrix, \mathbf{M} , can be multiplied to form a new matrix of the same size as \mathbf{M} , which is computed by $\mathbf{T} = a\mathbf{M} = [am_{ij}]$. \mathbf{T} and \mathbf{M} are of the same size, and these trivial rules apply: *i*) $0\mathbf{M} = \mathbf{0}$, *ii*) $1\mathbf{M} = \mathbf{M}$, *iii*) $a(b\mathbf{M}) = (ab)\mathbf{M}$, *iv*) $a\mathbf{0} = \mathbf{0}$, *v*) $(a + b)\mathbf{M} = a\mathbf{M} + b\mathbf{M}$, *vi*) $a(\mathbf{M} + \mathbf{N}) = a\mathbf{M} + a\mathbf{N}$.

Transpose of a Matrix

\mathbf{M}^T is the notation for the transpose of $\mathbf{M} = [m_{ij}]$, and the definition is $\mathbf{M}^T = [m_{ji}]$, i.e., the columns become rows and the rows become columns. For the transpose operator, we have: *i*) $(a\mathbf{M})^T = a\mathbf{M}^T$, *ii*) $(\mathbf{M} + \mathbf{N})^T = \mathbf{M}^T + \mathbf{N}^T$, *iii*) $(\mathbf{M}^T)^T = \mathbf{M}$, *iv*) $(\mathbf{MN})^T = \mathbf{N}^T\mathbf{M}^T$.

Trace of a Matrix

The trace of a matrix, denoted $\text{tr}(\mathbf{M})$, is simply the sum of the diagonal elements of a square matrix, as shown below:

$$\text{tr}(\mathbf{M}) = \sum_{i=0}^{n-1} m_{ii}. \quad (\text{A.24})$$

Matrix-Matrix Multiplication

This operation, denoted \mathbf{MN} between \mathbf{M} and \mathbf{N} , is defined only if \mathbf{M} is of size $p \times q$ and \mathbf{N} is of size $q \times r$, in which case the result, \mathbf{T} , becomes a $p \times r$ sized matrix. Mathematically, for these matrices the operation is as follows:

$$\begin{aligned} \mathbf{T} = \mathbf{MN} &= \begin{pmatrix} m_{00} & \cdots & m_{0,q-1} \\ \vdots & \ddots & \vdots \\ m_{p-1,0} & \cdots & m_{p-1,q-1} \end{pmatrix} \begin{pmatrix} n_{00} & \cdots & n_{0,r-1} \\ \vdots & \ddots & \vdots \\ n_{q-1,0} & \cdots & n_{q-1,r-1} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=0}^{q-1} m_{0,i}n_{i,0} & \cdots & \sum_{i=0}^{q-1} m_{0,i}n_{i,r-1} \\ \vdots & \ddots & \vdots \\ \sum_{i=0}^{q-1} m_{p-1,i}n_{i,0} & \cdots & \sum_{i=0}^{q-1} m_{p-1,i}n_{i,r-1} \end{pmatrix}. \end{aligned} \quad (\text{A.25})$$

In other words, each row of \mathbf{M} and column of \mathbf{N} are combined using a dot product, and the result placed in the corresponding row and column element. The elements of \mathbf{T} are computed as $t_{ij} = \sum_{k=0}^{q-1} m_{i,k}n_{k,j}$, which can also be expressed as $t_{ij} = \mathbf{m}_i \cdot \mathbf{n}_j$, that is, using dot product and the matrix-vector indexing. Note also that an $n \times 1$ matrix, $\mathbf{S} = (s_{00} \ s_{10} \ \cdots \ s_{n-1,0})^T$, can be considered an n -tuple vector. If seen as such, then matrix-vector multiplication, between \mathbf{M} ($p \times q$) and \mathbf{v} (q -tuple), can be derived from the definition of matrix-matrix multiplication. This is shown in Equation A.26, resulting in a new vector, \mathbf{w} :

$$\begin{aligned} \mathbf{w} = \mathbf{Mv} &= \begin{pmatrix} m_{00} & \cdots & m_{0,q-1} \\ \vdots & \ddots & \vdots \\ m_{p-1,0} & \cdots & m_{p-1,q-1} \end{pmatrix} \begin{pmatrix} v_0 \\ \vdots \\ v_{q-1} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=0}^{q-1} m_{0,k}v_k \\ \vdots \\ \sum_{k=0}^{q-1} m_{p-1,k}v_k \end{pmatrix} = \begin{pmatrix} \mathbf{m}_0 \cdot \mathbf{v} \\ \vdots \\ \mathbf{m}_{p-1} \cdot \mathbf{v} \end{pmatrix} = \begin{pmatrix} w_0 \\ \vdots \\ w_{p-1} \end{pmatrix}. \end{aligned} \quad (\text{A.26})$$

These three rules hold for the matrix-matrix multiplication: *i*) $(\mathbf{LM})\mathbf{N} = \mathbf{L}(\mathbf{MN})$, *ii*) $(\mathbf{L} + \mathbf{M})\mathbf{N} = \mathbf{LN} + \mathbf{MN}$, *iii*) $\mathbf{MI} = \mathbf{IM} = \mathbf{M}$. If the dimensions of the matrices are the same, then $\mathbf{MN} \neq \mathbf{NM}$, in general. This means that some pairs of matrices do commute, but usually they do not.

Determinant of a Matrix

The determinant is defined only for square matrices, and in the general case, the definition is recursive or defined via permutations [8]. Here, the focus will be on determinants for 2×2 and 3×3 matrices, since those are the ones most often needed in computer graphics.

The determinant of \mathbf{M} , written $|\mathbf{M}|$, for these matrices appears in Equation A.27 and A.28:

$$|\mathbf{M}| = \begin{vmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{vmatrix} = m_{00}m_{11} - m_{01}m_{10} \quad (\text{A.27})$$

$$\begin{aligned} |\mathbf{M}| &= \begin{vmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{vmatrix} \\ &= m_{00}m_{11}m_{22} + m_{01}m_{12}m_{20} + m_{02}m_{10}m_{21} \\ &\quad - m_{02}m_{11}m_{20} - m_{01}m_{10}m_{22} - m_{00}m_{12}m_{21} \end{aligned} \quad (\text{A.28})$$

In these two equations, a certain pattern can be distinguished: The positive terms are the elements multiplied in the diagonals from the top downward to the right and the negative terms are the elements in the diagonals from the top downward to the left, where a diagonal continues on the opposite side if an edge is crossed. Note that if the top row in \mathbf{M} is replaced by $\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z$, the middle row by $u_x \ u_y \ u_z$, and the bottom row by $v_x \ v_y \ v_z$, the cross product $\mathbf{u} \times \mathbf{v}$ is obtained according to Sarrus's scheme (Section A.2 on the cross product).

Another useful way to compute the determinant for 3×3 matrices is to use the dot and the cross product as in Equation A.29, which is reminiscent of column vector indexing:

$$|\mathbf{M}| = |\mathbf{m}_0 \ \mathbf{m}_1 \ \mathbf{m}_2| = |\mathbf{m}_x \ \mathbf{m}_y \ \mathbf{m}_z| = (\mathbf{m}_x \times \mathbf{m}_y) \cdot \mathbf{m}_z \quad (\text{A.29})$$

The following notation is also used for determinants:

$$|\mathbf{M}| = \det(\mathbf{M}) = \det(\mathbf{m}_x, \mathbf{m}_y, \mathbf{m}_z). \quad (\text{A.30})$$

Observe that the scalar triple product from Equation A.18 can be applied to Equation A.29; that is, if the vectors are rotated, the determinant remains unchanged, but changing the places of two vectors will change the sign of the determinant.

If $n \times n$ is the size of \mathbf{M} , then the following apply to determinant calculations: *i*) $|\mathbf{M}^{-1}| = 1/|\mathbf{M}|$, *ii*) $|\mathbf{MN}| = |\mathbf{M}| |\mathbf{N}|$, *iii*) $|a\mathbf{M}| = a^n |\mathbf{M}|$, *iv*) $|\mathbf{M}^T| = |\mathbf{M}|$. Also, if all elements of one row (or one column) are multiplied by a scalar, a , then $a|\mathbf{M}|$ is obtained, and if two rows (or columns) coincide (i.e., the cross product between them is zero), then $|\mathbf{M}| = 0$. The same result is obtained if any row or column is composed entirely of zeros.

The orientation of a basis can be determined via determinants. A basis is said to form a right-handed system, also called a positively oriented basis, if its determinant is positive. The standard basis has this property, since $|\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z| = (\mathbf{e}_x \times \mathbf{e}_y) \cdot \mathbf{e}_z = (0, 0, 1) \cdot \mathbf{e}_z = \mathbf{e}_z \cdot \mathbf{e}_z = 1 > 0$. If the determinant is negative, the basis is called negatively oriented or is said to be forming a left-handed system.

Some geometrical interpretations of the determinant are given in Section A.5.

Adjoins

The adjoint is another form of a matrix that can sometimes be useful. It is a matrix that is used for transforming surface normals. It is also a first step in computing the inverse of a matrix. We start by defining the subdeterminant (also called cofactor) $d_{ij}^{\mathbf{M}}$ of an $n \times n$ matrix \mathbf{M} as the determinant that is obtained by deleting row i and column j and then taking the determinant of the resulting $(n-1) \times (n-1)$ matrix. An example of computing the subdeterminant $d_{02}^{\mathbf{M}}$ of a 3×3 matrix is shown in Equation A.31:

$$d_{02}^{\mathbf{M}} = \begin{vmatrix} m_{10} & m_{11} \\ m_{20} & m_{21} \end{vmatrix}. \quad (\text{A.31})$$

For a 3×3 matrix, the adjoint is then

$$\text{adj}(\mathbf{M}) = \begin{pmatrix} d_{00} & -d_{10} & d_{20} \\ -d_{01} & d_{11} & -d_{21} \\ d_{02} & -d_{12} & d_{22} \end{pmatrix}, \quad (\text{A.32})$$

where we have left out the superscript \mathbf{M} of the subdeterminants for clarity. Note the signs and the order in which the subdeterminants appear. If we want to compute the adjoint \mathbf{A} of an arbitrary sized matrix \mathbf{M} , then the component at position (i, j) is

$$[a_{ij}] = [(-1)^{(i+j)} d_{ji}^{\mathbf{M}}]. \quad (\text{A.33})$$

Sometimes the adjoint has another definition than the one presented here, i.e., the adjoint of a matrix $\mathbf{M} = [m_{ij}]$ is denoted $\mathbf{M}^* = [\overline{m_{ji}}]$, where $\overline{m_{ji}}$ is the complex conjugate.

Inverse of a Matrix

The multiplicative inverse of a matrix, \mathbf{M} , denoted \mathbf{M}^{-1} , which is dealt with here, exists only for square matrices with $|\mathbf{M}| \neq 0$. If all elements of the matrix under consideration are real scalars, then it suffices to show that $\mathbf{M}\mathbf{N} = \mathbf{I}$ and $\mathbf{N}\mathbf{M} = \mathbf{I}$, where then $\mathbf{N} = \mathbf{M}^{-1}$. The problem can also be stated thus: If $\mathbf{u} = \mathbf{M}\mathbf{v}$ and a matrix \mathbf{N} exists such that $\mathbf{v} = \mathbf{N}\mathbf{u}$, then $\mathbf{N} = \mathbf{M}^{-1}$.

The inverse of a matrix can be computed either implicitly or explicitly. If the inverse is to be used several times, then it is more economical to compute \mathbf{M}^{-1} explicitly, i.e., to get a representation of the inverse as an array of $n \times n$ real numbers. On the other hand, if only a linear system of the type $\mathbf{u} = \mathbf{M}\mathbf{v}$ needs to be solved (for \mathbf{v}), then an implicit method, such as *Cramer's rule*, can be used. For a linear system of the type $\mathbf{M}\mathbf{v} = \mathbf{0}$, $|\mathbf{M}| = 0$ is a requirement if there is to be a solution, \mathbf{v} .

Using Cramer's rule to solve $\mathbf{u} = \mathbf{M}\mathbf{v}$ gives $\mathbf{v} = \mathbf{M}^{-1}\mathbf{u}$, but not \mathbf{M}^{-1} explicitly. Equation A.34 shows the general solution for the elements of \mathbf{v} :

$$v_i = \frac{d_i}{|\mathbf{M}|}, \quad (\text{A.34})$$

$$d_i = |\mathbf{m}_{,0} \quad \mathbf{m}_{,1} \quad \dots \quad \mathbf{m}_{,i-1} \quad \mathbf{u} \quad \mathbf{m}_{,i+1} \quad \dots \quad \mathbf{m}_{,n-1}|.$$

The terms d_i are thus computed as $|\mathbf{M}|$, except that column i is replaced by \mathbf{u} . For a 3×3 system, the solution obtained by Cramer's rule is presented below:

$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \frac{1}{|\mathbf{M}|} \begin{pmatrix} \det(\mathbf{u}, \mathbf{m}_y, \mathbf{m}_z) \\ \det(\mathbf{m}_x, \mathbf{u}, \mathbf{m}_z) \\ \det(\mathbf{m}_x, \mathbf{m}_y, \mathbf{u}) \end{pmatrix}. \quad (\text{A.35})$$

Many terms in this equation can be factorized using the scalar triple product rule and then reused for faster calculation.

For a 2×2 matrix, \mathbf{M} , the explicit solution is given by Equation A.36, and as can be seen, it is trivial to implement, since $|\mathbf{M}| = m_{00}m_{11} - m_{01}m_{10}$:

$$\mathbf{M}^{-1} = \frac{1}{|\mathbf{M}|} \begin{pmatrix} m_{11} & -m_{01} \\ -m_{10} & m_{00} \end{pmatrix}. \quad (\text{A.36})$$

For the general case, the adjoint from the previous section can be used:

$$\mathbf{M}^{-1} = \frac{1}{|\mathbf{M}|} \text{adj}(\mathbf{M}). \quad (\text{A.37})$$

In fact, this is Cramer's rule expressed differently to get the inverse matrix explicitly.

However, for larger sizes than 4×4 , there are no simple formulae, and Cramer's rule also becomes infeasible for matrices larger than 4×4 . Gaussian elimination is then the method of preference, and it can be used to solve for $\mathbf{u} = \mathbf{M}\mathbf{v} \Rightarrow \mathbf{v} = \mathbf{M}^{-1}\mathbf{u}$ —that is, an implicit solution, as is the case for Cramer's rule. However, Gaussian elimination can also be used to compute the matrix \mathbf{M}^{-1} explicitly. Consider the system in Equation A.38, where \mathbf{u} and \mathbf{v} are arbitrary vectors of the same dimension as \mathbf{M} and \mathbf{I} (the identity matrix):

$$\mathbf{M}\mathbf{u} = \mathbf{I}\mathbf{v}. \quad (\text{A.38})$$

Performing Gaussian elimination on this system until \mathbf{M} has been transformed into the identity matrix \mathbf{I} means that the right side identity matrix has become the inverse \mathbf{M}^{-1} . Thus, \mathbf{u} and \mathbf{v} are, in fact, not of any particular use; they are merely means for expressing Equation A.38 in a mathematically sound way.

LU decomposition is another method that can be used to compute the inverse efficiently. However, a discussion of Gaussian elimination and LU decomposition is beyond the scope of this text. Virtually any book on linear algebra [7, 8] or numerical methods books, such as the one by Press et al. [10], describe these methods.

Some important rules of computation for the inverse of matrices are:

$$i) (\mathbf{M}^{-1})^T = (\mathbf{M}^T)^{-1}, \quad ii) (\mathbf{M}\mathbf{N})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}.$$

Eigenvalue and Eigenvector Computation

The solution to the *eigenvalue* problem has a large range of uses. For example, one application area is the computation of tight bounding volumes. The problem is stated as follows:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \quad (\text{A.39})$$

where λ is a scalar. We use λ (even though this is not consistent with our notation) because this is what most texts use. The matrix \mathbf{A} has to be square (say of size $n \times n$), and $\mathbf{x} \neq \mathbf{0}$, then, if \mathbf{x} fulfills this equation, \mathbf{x} is said to be an *eigenvector* to \mathbf{A} , and λ is its belonging eigenvalue. Rearranging the terms of Equation A.39 yields Equation A.40:

$$(\lambda \mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}. \quad (\text{A.40})$$

This equation has a solution if and only if $p_A(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = 0$, where the function $p_A(\lambda)$ is called the characteristic polynomial to \mathbf{A} . The eigenvalues, $\lambda_0, \dots, \lambda_{n-1}$, are thus solutions to $p_A(\lambda) = 0$. Focus for a while on a particular eigenvalue λ_i to \mathbf{A} . Then \mathbf{x}_i is its corresponding eigenvector if $(\lambda_i \mathbf{I} - \mathbf{A})\mathbf{x}_i = \mathbf{0}$, which means that once the eigenvalues have been found, the eigenvectors can be found via Gaussian elimination.

Some theoretical results of great use are: *i*) $\text{tr}(\mathbf{A}) = \sum_{i=0}^{n-1} a_{ii} = \sum_{i=0}^{n-1} \lambda_i$, *ii*) $\det(\mathbf{A}) = \prod_{i=0}^{n-1} \lambda_i$, *iii*) if \mathbf{A} is real (consists of only real values) and is symmetric, i.e., $\mathbf{A} = \mathbf{A}^T$, then its eigenvalues are real and the different eigenvectors are orthogonal.

Orthogonal Matrices

Here, we will shed some light on the concept of an orthogonal matrix, its properties, and its characteristics. A square matrix, \mathbf{M} , with only real elements is orthogonal if and only if $\mathbf{M}\mathbf{M}^T = \mathbf{M}^T\mathbf{M} = \mathbf{I}$. That is, when multiplied by its transpose, it yields the identity matrix.

The orthogonality of a matrix, \mathbf{M} , has some significant implications such as: *i*) $|\mathbf{M}| = \pm 1$, *ii*) $\mathbf{M}^{-1} = \mathbf{M}^T$, *iii*) \mathbf{M}^T is also orthogonal, *iv*) $\|\mathbf{M}\mathbf{u}\| = \|\mathbf{u}\|$, *v*) $\mathbf{M}\mathbf{u} \perp \mathbf{M}\mathbf{v} \Leftrightarrow \mathbf{u} \perp \mathbf{v}$, *vi*) if \mathbf{M} and \mathbf{N} are orthogonal, so is $\mathbf{M}\mathbf{N}$.

The standard basis is orthonormal because the basis vectors are mutually orthogonal and of length one. Using the standard basis as a matrix, we can show that the matrix is orthogonal: $\mathbf{E} = (\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z) = \mathbf{I}$, and $\mathbf{I}^T\mathbf{I} = \mathbf{I}$.

Note there is a small mismatch here for how “orthogonal” is used in mathematics. An orthogonal basis consists of mutually perpendicular vectors that do not have to be normalized. An orthogonal matrix requires that all its rows (and columns) consist of normalized vectors, which are also mutually perpendicular. So, an orthogonal matrix is formed by an orthonormal, not orthogonal, basis. There is no such thing as an “orthonormal matrix.”

A.3.2 Change of Basis

Assume we have a vector, \mathbf{v} , in the standard basis described by the coordinate axes \mathbf{e}_x , \mathbf{e}_y , and \mathbf{e}_z . Furthermore, we have another coordinate system described by the arbitrary basis vectors \mathbf{f}_x , \mathbf{f}_y , and \mathbf{f}_z (which must be noncoplanar, i.e., $|\mathbf{f}_x \ \mathbf{f}_y \ \mathbf{f}_z| \neq 0$). How can \mathbf{v} be expressed uniquely in the basis described by \mathbf{f}_x , \mathbf{f}_y , and \mathbf{f}_z ? The solution

is given below, where \mathbf{w} is \mathbf{v} expressed in the new basis, described by \mathbf{F} [7]:

$$\begin{aligned} \mathbf{F}\mathbf{w} &= \begin{pmatrix} \mathbf{f}_x & \mathbf{f}_y & \mathbf{f}_z \end{pmatrix} \mathbf{w} = \mathbf{v} \\ &\iff \\ \mathbf{w} &= \mathbf{F}^{-1}\mathbf{v}. \end{aligned} \tag{A.41}$$

A special situation occurs if the matrix \mathbf{F} is orthogonal, which implies that the inverse is easily obtained by $\mathbf{F}^{-1} = \mathbf{F}^T$. Therefore, Equation A.41 simplifies to Equation A.42:

$$\mathbf{w} = \mathbf{F}^T \mathbf{v} = \begin{pmatrix} \mathbf{f}_x^T \\ \mathbf{f}_y^T \\ \mathbf{f}_z^T \end{pmatrix} \mathbf{v} = \begin{pmatrix} \mathbf{f}_x \cdot \mathbf{v} \\ \mathbf{f}_y \cdot \mathbf{v} \\ \mathbf{f}_z \cdot \mathbf{v} \end{pmatrix}. \tag{A.42}$$

Orthogonal coordinate system changes are the most common ones in the context of computer graphics.

A.4 Homogeneous Notation

This section is probably the most important in this chapter, since it influences many areas of computer graphics and is almost never treated in a common linear algebra book.

A point describes a location in space, while a vector describes a direction and has no location. Using 3×3 matrices (or 2×2 for two dimensions), it is possible to perform linear transformations such as rotations, scalings, and shears on coordinates. However, translation cannot be performed using such matrices. This lack is unimportant for vectors, for which translation has no meaning, but translation is meaningful for points.

Homogeneous notation is useful for transforming both vectors and points, and includes the ability to perform translation only on points. It augments 3×3 matrices to the size of 4×4 , and three-dimensional points and vectors get one more element. So, a homogeneous vector is $\mathbf{p} = (p_x, p_y, p_z, p_w)$. As will soon be made clear, $p_w = 1$ for points, and $p_w = 0$ for vectors. For projections, other values can be used for p_w . When $p_w \neq 0$ and $p_w \neq 1$, then the actual point is obtained through *homogenization*, where all components are divided by p_w . This means the point $(p_x/p_w, p_y/p_w, p_z/p_w, 1)$ is obtained.

Equation A.43 shows how a 3×3 matrix, \mathbf{M} , is augmented (in the simplest case) into the homogeneous form:

$$\mathbf{M}_{4 \times 4} = \begin{pmatrix} m_{00} & m_{01} & m_{02} & 0 \\ m_{10} & m_{11} & m_{12} & 0 \\ m_{20} & m_{21} & m_{22} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.43}$$

Rotation, scaling, and shear matrices can replace \mathbf{M} in this equation, and affect both vectors and points, as they should. A translation, however, uses the additional

elements of the augmented matrix to handle translations. A typical translation matrix, \mathbf{T} , which translates a point by a vector, \mathbf{t} , is shown in Equation A.44:

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (\text{A.44})$$

The combination of a linear transform followed by a translation is called an *affine transform*.

It is quickly verified that a vector $\mathbf{v} = (v_x, v_y, v_z, 0)$ is unaffected by the $\mathbf{T}\mathbf{v}$ transform because its last element is 0. If the point $\mathbf{p} = (p_x, p_y, p_z, 1)$ is transformed as $\mathbf{T}\mathbf{p}$, the result is $(p_x + t_x, p_y + t_y, p_z + t_z, 1)$, i.e., \mathbf{p} translated by \mathbf{t} .

Matrix-matrix multiplications (and thus concatenations of homogeneous matrices) and matrix-vector multiplications are carried out precisely as usual. In Chapter 4.5, many kinds of different homogeneous transforms will be introduced and thoroughly dissected.

A.5 Geometry

This section is concerned with presenting some useful geometrical techniques that are used extensively in, for example, Chapter 22.

A.5.1 Lines

Two-Dimensional Lines

For two-dimensional lines, there are two main mathematical descriptions: the implicit form and the explicit form. The latter is parametric, and a typical equation for it is Equation A.45:

$$\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}. \quad (\text{A.45})$$

Here, \mathbf{o} is a point on the line, \mathbf{d} is the direction vector of the line, and t is a parameter that can be used to generate different points, \mathbf{r} , on the line. We name the points \mathbf{r} here because in other chapters the line is usually referred to as a ray, in which $t \geq 0$ and \mathbf{o} is called the *origin*.

The implicit form is different in that it cannot generate points explicitly. Assume, instead, that on our line of interest, called L , any point can be described by $\mathbf{p} = (p_x, p_y) = (x, y)$, where the latter coordinate notation is used only in Equation A.46, because this is the line equation form that most people learn first:

$$ax + by + c = 0. \quad (\text{A.46})$$

For \mathbf{p} to lie on the line L , it must fulfill this equation. Now, if the constants a and b are combined into a vector $\mathbf{n} = (n_x, n_y) = (a, b)$, then Equation A.46 can be

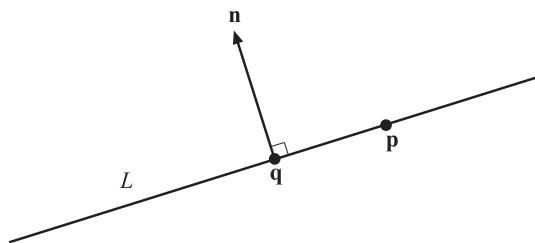


Figure A.5. Notation used for the implicit form of a two-dimensional line equation. L is the line, \mathbf{p} and \mathbf{q} are points on the line, and \mathbf{n} is a vector perpendicular to the direction of L .

expressed using the dot product as

$$\mathbf{n} \cdot \mathbf{p} + c = 0. \quad (\text{A.47})$$

A common task is, naturally, to compute the constants of a line. Looking at Figure A.5 and using its notation, we may see that one way of describing L would be $\mathbf{n} \cdot (\mathbf{p} - \mathbf{q}) = 0$, where \mathbf{q} is another point on the line. Thus, \mathbf{n} must be perpendicular to the line direction, i.e., $\mathbf{n} = (\mathbf{p} - \mathbf{q})^\perp = (-(p_y - q_y), p_x - q_x) = (a, b)$, and $c = -\mathbf{q} \cdot \mathbf{n}$. Then, given two points on L , $\mathbf{n} = (a, b)$ and c can be calculated.

Rewriting Equation A.47 as a function, $f(\mathbf{p}) = \mathbf{n} \cdot \mathbf{p} + c$, allows some key properties to be derived. Nothing is now assumed about \mathbf{p} ; it can be any point in the two-dimensional plane. First, if again $\mathbf{q} \in L$, the following holds:

1. $f(\mathbf{p}) = 0 \iff \mathbf{p} \in L$.
2. $f(\mathbf{p}) > 0 \iff \mathbf{p}$ lies on the same side of the line as the point $\mathbf{q} + \mathbf{n}$.
3. $f(\mathbf{p}) < 0 \iff \mathbf{p}$ lies on the same side of the line as the point $\mathbf{q} - \mathbf{n}$.

This test is usually called a *half-plane test* and is used in, for example, some point-in-polygon routines. Second, $f(\mathbf{p})$ happens to be a measure for the perpendicular distance from \mathbf{p} to L . In fact, it turns out that the *signed distance*, f_s , is found by using Equation A.48:

$$f_s(\mathbf{p}) = \frac{f(\mathbf{p})}{\|\mathbf{n}\|}. \quad (\text{A.48})$$

If $\|\mathbf{n}\| = 1$, that is, \mathbf{n} is normalized, then $f_s(\mathbf{p}) = f(\mathbf{p})$.

Three-Dimensional Lines

In three dimensions, the implicit form of a line is often expressed as the intersection between two nonparallel planes. An explicit form is shown in Equation A.49. This equation is heavily used in computing the intersection between a line (or ray) and a three-dimensional surface:

$$\mathbf{r}(t) = \mathbf{o} + t\mathbf{d}. \quad (\text{A.49})$$

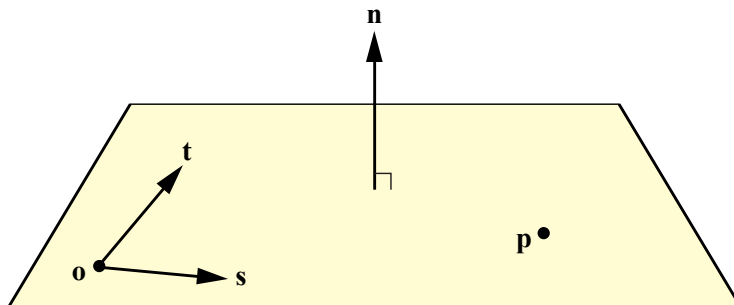


Figure A.6. A plane can be defined by $\mathbf{p}(u, v) = \mathbf{o} + us + vt$, where \mathbf{o} is a point in the plane, and \mathbf{s} and \mathbf{t} are non-parallel vectors in the plane. Alternatively, the plane can be expressed as $\mathbf{n} \cdot \mathbf{p} + d = 0$, where \mathbf{n} is the normal of the plane and \mathbf{p} is any point in the plane. The value of d can be computed as $d = -\mathbf{n} \cdot \mathbf{x}$, where \mathbf{x} is an arbitrary point in the plane.

This is exactly the same equation used for the two-dimensional line, except that the points and the direction vector are now three dimensional.

The distance from a point \mathbf{p} to $\mathbf{r}(t)$ can be computed by first projecting \mathbf{p} onto $\mathbf{r}(t)$, using Equation A.17. Assuming \mathbf{d} is normalized, this is done by computing the vector $\mathbf{w} = ((\mathbf{p} - \mathbf{o}) \cdot \mathbf{d})\mathbf{d}$. Then the sought-for distance is $\|(\mathbf{p} - \mathbf{o}) - \mathbf{w}\|$.

A.5.2 Planes

Since the three-dimensional plane is a natural extension of the two-dimensional line, it can be expected that the plane has similar properties as the line. In fact, any hyperplane, which is the name for a plane of an arbitrary dimension, has these similar characteristics. Here, the discussion will be limited to the three-dimensional plane.

First, the plane can be described in both explicit and implicit form. Equation A.50 shows the explicit form of the three-dimensional plane:

$$\mathbf{p}(u, v) = \mathbf{o} + us + vt. \quad (\text{A.50})$$

Here, \mathbf{o} is a point lying on the plane, \mathbf{s} and \mathbf{t} are direction vectors that span the plane (i.e., they are noncollinear), and u and v are parameters that generate different points on the plane. The normal of this plane is $\mathbf{s} \times \mathbf{t}$. This is illustrated in Figure A.6.

The implicit equation for the plane, called π , shown in Equation A.51, is identical to Equation A.47, with the exception that the plane equation is augmented with an extra dimension, and c has been renamed to d :

$$\mathbf{n} \cdot \mathbf{p} + d = 0. \quad (\text{A.51})$$

Again, \mathbf{n} is the normal of the plane, \mathbf{p} is any point on the plane, and d is a constant that determines part of the position of the plane. As was shown previously, the normal

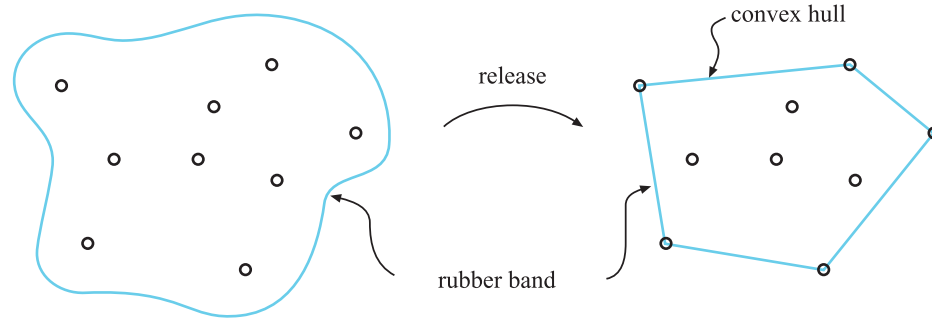


Figure A.7. The rubberband/nail scheme in action. The left figure shows the rubber band before it has been released and the right figure shows the result after. Then the rubber band is a representation of the convex hull of the nails.

can be computed using two direction vectors, but it can naturally also be obtained from three noncollinear points, \mathbf{u} , \mathbf{v} , and \mathbf{w} , lying on the plane, as $\mathbf{n} = (\mathbf{u} - \mathbf{w}) \times (\mathbf{v} - \mathbf{w})$. Given \mathbf{n} and a point \mathbf{q} on the plane, the constant is computed as $d = -\mathbf{n} \cdot \mathbf{q}$. The half-plane test is equally valid. By denoting $f(\mathbf{p}) = \mathbf{n} \cdot \mathbf{p} + d$, the same conclusions can be drawn as for the two-dimensional line; that is,

1. $f(\mathbf{p}) = 0 \iff \mathbf{p} \in \pi$.
2. $f(\mathbf{p}) > 0 \iff \mathbf{p}$ lies on the same side of the plane as the point $\mathbf{q} + \mathbf{n}$.
3. $f(\mathbf{p}) < 0 \iff \mathbf{p}$ lies on the same side of the plane as the point $\mathbf{q} - \mathbf{n}$.

The signed distance from an arbitrary point \mathbf{p} to π is obtained by exchanging the two-dimensional parts of Equation A.48 for their three-dimensional counterparts for the plane. Also, it is easy to interpret the meaning of d using the signed distance formula. This requires inserting the origin, $\mathbf{0}$, into that formula; $f_s(\mathbf{0}) = d$, which means that d is the shortest (signed) distance from the origin to the plane.

A.5.3 Convex Hull

For a set of points, the *convex hull* is defined as the smallest set that satisfies the condition that the straight line between any two points in the set is entirely included in the set as well. This holds for any dimension.

In two dimensions, the construction of the convex hull is intuitively illustrated by the rubberband/nail scheme. Imagine that the points are nails in a table, and that a rubber band is held in such a way that its interior includes all nails. The rubber band is then released, and now the rubber band is, in fact, the convex hull of the nails. This is illustrated in Figure A.7.

The convex hull has many areas of use; the construction of bounding volumes is one. The convex hull is, by its definition, the smallest convex volume for a set of

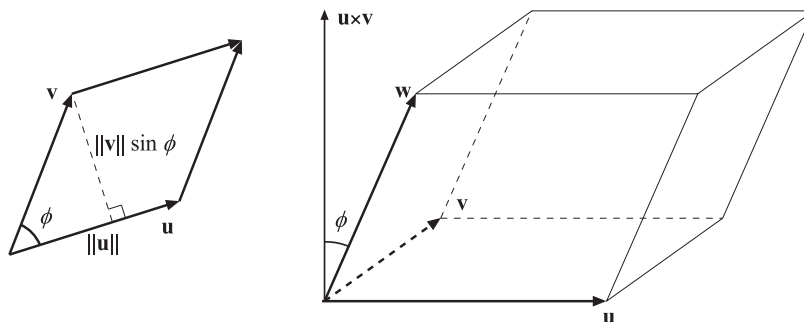


Figure A.8. Left figure: a parallelogram whose area is computed by multiplying the length of the base ($\|\mathbf{u}\|$) with the height ($\|\mathbf{v}\| \sin \phi$). Right figure: The volume of a parallelepiped is given by $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

points, and is therefore attractive for those computations. Algorithms for computing the convex hull in two and three dimensions can be found in, for example, books by de Berg et al. [3] or O'Rourke [9]. A fast algorithm, called *QuickHull*, is presented by Barber et al. [1, 6].

A.5.4 Miscellaneous

Area Calculation

A parallelogram defined by two vectors, \mathbf{u} and \mathbf{v} , starting at the origin, has the area $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \phi$, as shown in Figure A.8.

If the parallelogram is divided by a line from \mathbf{u} to \mathbf{v} , two triangles are formed. This means that the area of each triangle is half the area of the parallelogram. So, if a triangle is given by three points, say \mathbf{p} , \mathbf{q} , and \mathbf{r} , its area is

$$\text{Area}(\triangle \mathbf{pqr}) = \frac{1}{2} \|(\mathbf{p} - \mathbf{r}) \times (\mathbf{q} - \mathbf{r})\|. \quad (\text{A.52})$$

The signed area of a general two-dimensional polygon can be computed by [9, 11]

$$\text{Area}(P) = \frac{1}{2} \sum_{i=0}^{n-1} (x_i y_{i+1} - y_i x_{i+1}), \quad (\text{A.53})$$

where n is the number of vertices, and where i is modulo n , so that $x_n = x_0$ and $y_n = y_0$. This area can be computed with fewer multiplies and potentially more accuracy

(though needing a wider range of indices for each term) by the following formula [13]:

$$\text{Area}(P) = \frac{1}{2} \sum_{i=0}^{n-1} (x_i(y_{i+1} - y_{i-1})). \quad (\text{A.54})$$

The sign of the area is related to the order in which the outline of the polygon is traversed; positive means counterclockwise in a right-handed coordinate system.

Volume Calculation

The scalar triple product, from Equation A.18, is sometimes also called the volume formula. Three vectors, \mathbf{u} , \mathbf{v} and \mathbf{w} , starting at the origin, form a solid, called a parallelepiped, whose volume is given by the equation that follows. The volume and the notation are depicted in Figure A.8:

$$\text{Volume}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}) \quad (\text{A.55})$$

This is a positive value only if the vectors form a positively oriented basis. The formula intuitively explains why the determinant of three vectors is zero if the vectors do not span the entire space \mathbb{R}^3 : In that case, the volume is zero, meaning that the vectors must lie in the same plane (or one or more of them may be the zero vector).

Further Reading and Resources

The reader is directed to <http://www.immersivemath.com> by Ström et al. [12] for a more thorough treatment of linear algebra. Beyond text, it also includes interactive illustrations to play with and learn from. Much of the material in this appendix is included in the *CRC Standard Mathematical Tables and Formulas* [14], as well as a huge amount of other mathematical knowledge.



Appendix B

Trigonometry

“Life is good for only two things, discovering mathematics and teaching mathematics.”

—Siméon Poisson

This appendix is intended to be a reference to some simple laws of trigonometry as well as some more sophisticated ones. The laws of trigonometry are particularly important tools in computer graphics. One example of their usefulness is that they provide ways to simplify equations and thereby increase speed.

B.1 Definitions

According to Figure B.1, where $\mathbf{p} = (p_x, p_y)$ is a unit vector, i.e., $\|\mathbf{p}\| = 1$, the fundamental trigonometric functions, sin, cos, and tan, are defined by Equation B.1.

Fundamental trigonometric functions :

$$\sin \phi = p_y$$

$$\cos \phi = p_x$$

$$\tan \phi = \frac{\sin \phi}{\cos \phi} = \frac{p_y}{p_x}$$

(B.1)

The sin, cos, and tan functions can be expanded into MacLaurin series, as shown in Equation B.2. MacLaurin series are a special case of the more general Taylor series. A Taylor series is an expansion about an arbitrary point, while a MacLaurin series always is developed around $x = 0$. MacLaurin series are beneficial because they clarify the origins of some of the derivatives (shown in Equation set B.4).

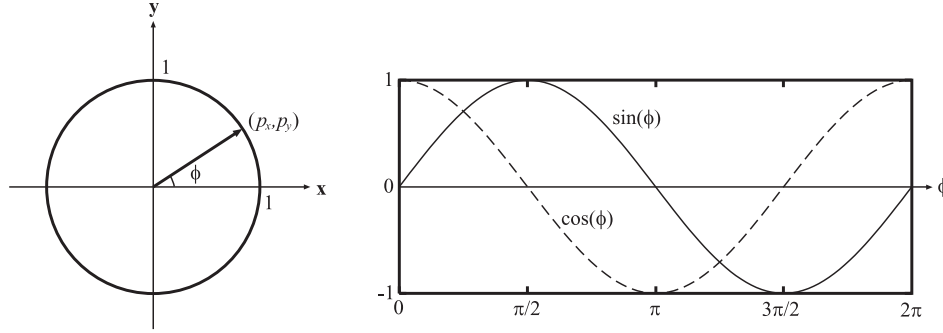


Figure B.1. The geometry for the definition of the sin, cos, and tan functions is shown to the left. The right-hand part of the figure shows $p_x = \cos \phi$ and $p_y = \sin \phi$, which together traces out the circle.

MacLaurin series :

$$\sin \phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \frac{\phi^7}{7!} + \dots + (-1)^n \frac{\phi^{2n+1}}{(2n+1)!} + \dots$$

$$\cos \phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \frac{\phi^6}{6!} + \dots + (-1)^n \frac{\phi^{2n}}{(2n)!} + \dots$$

$$\tan \phi = \phi + \frac{\phi^3}{3} + \frac{2\phi^5}{15} + \dots + (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} \phi^{2n-1} + \dots$$

(B.2)

The two first series hold for $-\infty < \phi < \infty$, the last one for $-\pi/2 < \phi < \pi/2$ and B_n is the n th Bernoulli number. The Bernoulli numbers can be generated with a recursive formula, where $B_0 = 1$ and then for $k > 1$, $\sum_{j=0}^{k-1} \binom{k}{j} B_j = 0$.

The inverses of the trigonometric functions, arcsin, arccos, and arctan, are defined as in Equation B.3.

Inverses of trigonometric functions :

$$p_y = \sin \phi \Leftrightarrow \phi = \arcsin p_y, \quad -1 \leq p_y \leq 1, \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$$

$$p_x = \cos \phi \Leftrightarrow \phi = \arccos p_x, \quad -1 \leq p_x \leq 1, \quad 0 \leq \phi \leq \pi$$

$$\frac{p_y}{p_x} = \tan \phi \Leftrightarrow \phi = \arctan \frac{p_y}{p_x}, \quad -\infty \leq \frac{p_y}{p_x} \leq \infty, \quad -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$$

(B.3)

The derivatives of the trigonometric functions and their inverses are summarized below.

Trigonometric derivatives :

$$\frac{d \sin \phi}{d\phi} = \cos \phi$$

$$\frac{d \cos \phi}{d\phi} = -\sin \phi$$

$$\frac{d \tan \phi}{d\phi} = \frac{1}{\cos^2 \phi} = 1 + \tan^2 \phi$$

$$\frac{d \arcsin t}{dt} = \frac{1}{\sqrt{1-t^2}}$$

$$\frac{d \arccos t}{dt} = -\frac{1}{\sqrt{1-t^2}}$$

$$\frac{d \arctan t}{dt} = \frac{1}{1+t^2}$$

(B.4)

B.2 Trigonometric Laws and Formulae

We begin with some fundamental laws about right triangles. To use the notation from Figure B.2, the following laws apply:

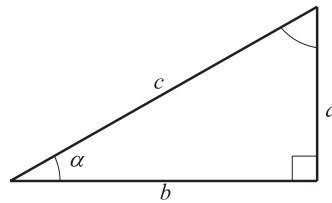


Figure B.2. A right triangle and its notation.

<p>Right triangle laws :</p> $\sin \alpha = \frac{a}{c}$ $\cos \alpha = \frac{b}{c}$ $\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{a}{b}$	(B.5)
---	-------

Pythagorean relation : $c^2 = a^2 + b^2$	(B.6)
---	-------

For arbitrarily angled triangles, the following well-known rules are valid, using the notation from Figure B.3.

<p>Law of sines : $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$</p> <p>Law of cosines : $c^2 = a^2 + b^2 - 2ab \cos \gamma$</p> <p>Law of tangents : $\frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$</p>	(B.7)
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Named after their inventors, the following two formulae are also valid for arbitrarily angled triangles.

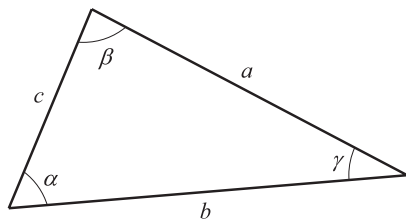


Figure B.3. An arbitrarily angled triangle and its notation.

$$\begin{array}{l}
 \text{Newton's formula : } \frac{b+c}{a} = \frac{\cos \frac{\beta-\gamma}{2}}{\sin \frac{\alpha}{2}} \\
 \text{Mollweide's formula : } \frac{b-c}{a} = \frac{\sin \frac{\beta-\gamma}{2}}{\cos \frac{\alpha}{2}}
 \end{array}
 \tag{B.8}$$

The definition of the trigonometric functions (Equation B.1) together with the Pythagorean relation (Equation B.6) gives the *trigonometric identity*:

$$\text{Trigonometric identity : } \cos^2 \phi + \sin^2 \phi = 1
 \tag{B.9}$$

Here follow some laws that can be exploited to simplify equations and thereby make their implementation more efficient.

$$\begin{array}{l}
 \text{Double angle relations :} \\
 \sin 2\phi = 2 \sin \phi \cos \phi = \frac{2 \tan \phi}{1 + \tan^2 \phi} \\
 \cos 2\phi = \cos^2 \phi - \sin^2 \phi = 1 - 2 \sin^2 \phi = 2 \cos^2 \phi - 1 = \frac{1 - \tan^2 \phi}{1 + \tan^2 \phi} \\
 \tan 2\phi = \frac{2 \tan \phi}{1 - \tan^2 \phi}
 \end{array}
 \tag{B.10}$$

Extensions of these laws are called the *multiple angle relations*, shown below.

$$\begin{array}{l}
 \text{Multiple angle relations :} \\
 \sin(n\phi) = 2 \sin((n-1)\phi) \cos \phi - \sin((n-2)\phi) \\
 \cos(n\phi) = 2 \cos((n-1)\phi) \cos \phi - \cos((n-2)\phi) \\
 \tan(n\phi) = \frac{\tan((n-1)\phi) + \tan \phi}{1 - \tan((n-1)\phi) \tan \phi}
 \end{array}
 \tag{B.11}$$

Equations B.12 and B.13 show a collection of laws that we call the *angle sum* and *angle difference relations*.

Angle sum relations :

$$\begin{aligned}\sin(\phi + \rho) &= \sin \phi \cos \rho + \cos \phi \sin \rho \\ \cos(\phi + \rho) &= \cos \phi \cos \rho - \sin \phi \sin \rho \\ \tan(\phi + \rho) &= \frac{\tan \phi + \tan \rho}{1 - \tan \phi \tan \rho}\end{aligned}\tag{B.12}$$

Angle difference relations :

$$\begin{aligned}\sin(\phi - \rho) &= \sin \phi \cos \rho - \cos \phi \sin \rho \\ \cos(\phi - \rho) &= \cos \phi \cos \rho + \sin \phi \sin \rho \\ \tan(\phi - \rho) &= \frac{\tan \phi - \tan \rho}{1 + \tan \phi \tan \rho}\end{aligned}\tag{B.13}$$

Next follow the *product relations*.

Product relations :

$$\begin{aligned}\sin \phi \sin \rho &= \frac{1}{2}(\cos(\phi - \rho) - \cos(\phi + \rho)) \\ \cos \phi \cos \rho &= \frac{1}{2}(\cos(\phi - \rho) + \cos(\phi + \rho)) \\ \sin \phi \cos \rho &= \frac{1}{2}(\sin(\phi - \rho) + \sin(\phi + \rho))\end{aligned}\tag{B.14}$$

The formulae in Equations B.15 and B.16 go under the names *function sums and differences* and *half-angle relations*.

Function sums and differences :

$$\sin \phi + \sin \rho = 2 \sin \frac{\phi + \rho}{2} \cos \frac{\phi - \rho}{2}$$

$$\cos \phi + \cos \rho = 2 \cos \frac{\phi + \rho}{2} \cos \frac{\phi - \rho}{2}$$

$$\tan \phi + \tan \rho = \frac{\sin(\phi + \rho)}{\cos \phi \cos \rho}$$

$$\sin \phi - \sin \rho = 2 \cos \frac{\phi + \rho}{2} \sin \frac{\phi - \rho}{2}$$

$$\cos \phi - \cos \rho = -2 \sin \frac{\phi + \rho}{2} \sin \frac{\phi - \rho}{2}$$

$$\tan \phi - \tan \rho = \frac{\sin(\phi - \rho)}{\cos \phi \cos \rho}$$

(B.15)

Half-angle relations :

$$\sin \frac{\phi}{2} = \pm \sqrt{\frac{1 - \cos \phi}{2}}$$

$$\cos \frac{\phi}{2} = \pm \sqrt{\frac{1 + \cos \phi}{2}}$$

$$\tan \frac{\phi}{2} = \pm \sqrt{\frac{1 - \cos \phi}{1 + \cos \phi}} = \frac{1 - \cos \phi}{\sin \phi} = \frac{\sin \phi}{1 + \cos \phi}$$

(B.16)

Further Reading and Resources

The first chapter of *Graphics Gems* [5] provides other geometric relationships that are useful in computer graphics. The classic reference book *CRC Standard Mathematical Tables and Formulas* [14] includes the formulae in this appendix and much more.



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